AFOSR 1742

A TECHNIQUE FOR THE SYNTHESIS OF LINEAR, NONSTATIONARY FEEDBACK SYSTEMS

PART II

THE SYNTHESIS PROBLEM

A. R. Stubberud

(Report No. 61-71)
University of California
Department of Engineering
Los Angeles, California

October, 1961

CONTRACT NO. AF 49(638)-438

MECHANICS DIVISION
AIR FORCE OFFICE OF SCIENTIFIC RESEARCH
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FOREWORD

The research described in this report, *A Technique for the Synthesis of Linear, Nonstationary Feedback Systems, Part II--The Synthesis Problem,* by A. R. Stubberud was carried out under the technical direction of C. T. Leondes and Gerald Estrin and is part of the continuing program in Adaptive Control Systems.

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DEPARTMENT OF ENGINEERING
LOS ANGELES, CALIFORNIA

TABLE OF CONTENTS

LIST OF FIGURES

A TECHNIQUE FOR THE SYNTHESIS OF LINEAR, NONSTATIONARY FEEDBACK SYSTEMS
PART II. THE SYNTHESIS PROBLEM

A. R. Stubberud

Summary: In this paper and a companion paper a technique for the synthesis of linear, nonstationary feedback systems is presented. This paper deals with the synthesis problem, i.e., a method is developed whereby a given weighting function can be synthesized as a feedback system. The problem of synthesizing a nonstationary weighting function as a feedback system when constrained by a fixed plant is also solved. A differential equation algebra which allows the solution of these problems is developed.

INTRODUCTION

In a companion paper 18 a technique for synthesizing linear, nonstationary feedback systems which parallel the synthesis technique proposed by Guillemin, 1, for linear, stationary feedback systems was proposed. In the companion paper the approximation problem was discussed. In this paper the synthesis problem will be discussed. An algebra of linear differential equations is first developed, and this algebra is then used to show that a given weighting function can always be synthesized as a feedback system.

AN OPERATOR ALGEBRA FOR DIFFERENTIAL EQUATIONS

Once the weighting function of a system has been determined by either the technique described in a companion paper 18 or by one of the other known techniques, the next step in the development is to realize this system in the form of a feedback system. First, however, the differential equation of which the weighting function is a solution must be determined. A technique for generating a differential equation from its weighting function has been discussed 2, 4, elsewhere; therefore only the necessary steps are repeated here.

If a weighting function has the form

$$W(t,\tau) = \sum_{i=1}^{n} \beta_i(\tau)\, q_i(t) + b_n(t)\, \delta(t-\tau) \qquad t \geq \tau$$

$$= 0 \qquad t < \tau \tag{1}$$

where the $\beta_i(\tau)$ and $q_i(t)$ have a sufficient number of derivatives, then $W(t,\tau)$ satisfies the differential equation

$$\sum_{i=o}^{n} a_i(t) \frac{\partial^i}{\partial t^i}\Big[W(t,\tau)\Big] = \sum_{j=o}^{n} b_j(t) \frac{\partial^j}{\partial t^j}\Big[\delta(t-\tau)\Big] \tag{2}$$

where $a_n(t) = 1$ and the $a_i(t)$ and $b_j(t)$ are continuous functions of time. The $a_i(t)$ and $b_j(t)$ of the differential equation can then be determined from the known $W(t,\tau)$ as follows. The $a_i(t)$ are obtained immediately as solutions of the n simultaneous equations.

$$\sum_{i=o}^{n-1} a_i(t) \frac{d^i q_j(t)}{dt^i} = - \frac{d^n q_j(t)}{dt^n} \qquad j = 1, 2, \ldots, n \tag{3}$$

This set of equations always has a unique solution since the determinant of the equations is the Wronskian of the $q_j(t)$ which are linearly independent functions. After the $a_i(t)$ have been determined, the $b_j(t)$ are determined from the equations

$$b_{n-i}(t) = \sum_{k=o}^{i} \sum_{s=o}^{i-k} \binom{n+s-i}{n-i} a_{n-i+k+s}(t) \frac{d^s F_k(t)}{dt^s} \tag{4}$$

$$i = 1, 2, \ldots, n$$

where $\binom{n+s-i}{n-i}$ is the binomial coefficient and the $F_k(t)$ are given by the relationships:

$$F_o(t) = b_n(t)$$

$$F_k(t) = \left\{ \frac{\partial^{k-1}}{\partial t^{k-1}} \left[W(t,\tau) \right] \right\} \Bigg|_{\tau = t^-} \qquad k = 1, 2, \ldots, n \tag{5}$$

Thus application of Equations (3), (4), and (5) allow generation of a differential equation from its corresponding weighting function. This step is necessary since from this point on all manipulations are done on the differential equations which correspond to the elements in the system. (It should be kept in mind that whenever possible it is desirable to work with the corresponding weighting functions; these are in general unknown, hence the more general differential equation approach is used.)

To aid in the development in the next section it is desirable to develop an operator algebra which will simplify many of the manipulations which are found necessary. Since a linear differential equation of the form

$$\sum_{i=o}^{n} a_i(t) \frac{d^i y}{dt^i} = \sum_{j=o}^{n} b_j(t) \frac{d^j x}{dt^j} \tag{6}$$

is a linear transformation of x into y, the algebra is an algebra of linear transformations. The transformations (linear differential equations) which are considered will all have the form of Equation (6) where some of the $a_i(t)$ and $b_j(t)$ may be zero. In the following development, capital letters (A, B, C...) are used to denote differential equations and represent equations of the form of Equation (6). In addition, all initial conditions are assumed to be zero.

The algebra of differential equations involves three operations:

(1) Addition of two differential equations, i.e.,

$$A + B = C \tag{7}$$

(2) Multiplication of differential equation by a scalar, i.e.,

$$p(t)\ A = B \tag{8}$$

$$\text{or } A\ p(t) = C \tag{9}$$

(Clearly if p(t) is a constant, B = C.)

(3) Multiplication of two differential equations, i.e.,

$$A\ B = C \tag{10}$$

These operations are indicated in block diagram form in Figure 1. Obviously, in order for this algebra to be useful, each of these operations must be defined. In addition the following useful properties of this algebra will be defined:

(1) A unity element

(2) A zero element

(3) An additive inverse

(4) A multiplicative inverse

Multiplication of Two Differential Equations. It is necessary to define multiplication first since it is used in defining addition. It is obvious from Figure 1(d) that multiplication is identical to convolution if A and B represent weighting functions.

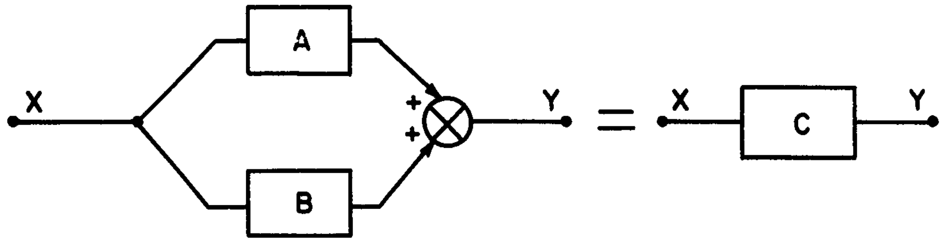
Any differential equation of the form in Equation (6) can be divided into two parts: a differential operator and an integral operator as indicated in Figure 2. The relationships between the variables x, y and z are, in terms of the notation of Equation (6):

$$z = \sum_{j=o}^{n} b_j(t) \frac{d^j x}{dt^j} \tag{11}$$

$$\sum_{i=o}^{n} a_i(t) \frac{d^i y}{dt^i} = z \tag{12}$$

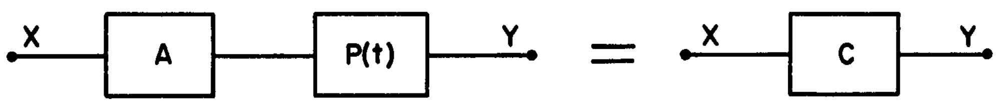
Now if two differential equations are multiplied together, this operation can be represented by Figure 3(a). The equations which define the relationships between the variables in this system are:

$$\sum_{j=o}^{n} b_j(t) \frac{d^j x}{dt^j} = x_1 = \sum_{i=o}^{n} a_i(t) \frac{d^i y}{dt^i} \tag{13}$$

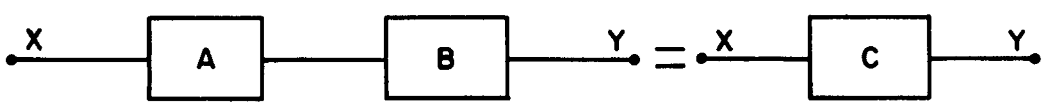


(a) ADDITION OF TWO DIFFERENTIAL EQUATIONS

(b) PRE-MULTIPLICATION OF A DIFFERENTIAL EQUATION BY A SCALAR



(c) POST-MULTIPLICATION OF A DIFFERENTIAL EQUATION BY A SCALAR



(d) MULTIPLICATION OF TWO DIFFERENTIAL EQUATIONS

OPERATIONS OF TRANSFORMATION ALGEBRA

FIGURE 1

$$\sum_{s=o}^{m} f_s(t) \frac{d^s y}{dt^s} = y_1 = \sum_{k=o}^{m} c_k(t) \frac{d^k z}{dt^k} \tag{14}$$

In Figure 3(b) the two inner operators in Figure 3(a) have been "interchanged". In general these operators are not commutable, therefore

$$\mathrm{Diff}_2 \neq \mathrm{Diff}_3$$

$$\mathrm{Int}_1 \neq \mathrm{Int}_3$$

It is necessary at this point to determine the relationships between x_1, W and y_1 in terms of the parameters of Equations (13) and (14). Assume that these relationships can be written

$$\sum_{r=o}^{m} g_r(t) \frac{d^r x_1}{dt^r} = W \tag{15}$$

and

$$\sum_{s=o}^{n} h_s(t) \frac{d^s y_1}{dt^s} = W \tag{16}$$

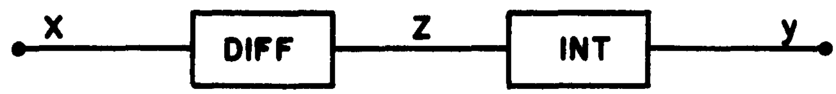
where $g_r(t)$ and $h_s(t)$ are as yet unknown coefficients. Substituting the values for x_1 and y_1, each in terms of y, into Equations (15) and (16) produces the relationship:

$$\sum_{\alpha=o}^{m} \sum_{i=o}^{n} \sum_{c=o} \binom{\alpha}{c} g_\alpha(t) \frac{d^{(\alpha-c)} a_i(t)}{dt^{(\alpha-c)}} \frac{d^{(i+c)} y}{dt^{(i+c)}} =$$

$$\sum_{\beta=o}^{n} \sum_{1=o}^{m} \sum_{\delta=o} \binom{\beta}{s} h_\beta(t) \frac{d^{(\beta-\delta)} f_1(t)}{dt^{(\beta-\delta)}} \frac{d^{(1+\delta)} y}{dt^{(1+\delta)}} \tag{17}$$

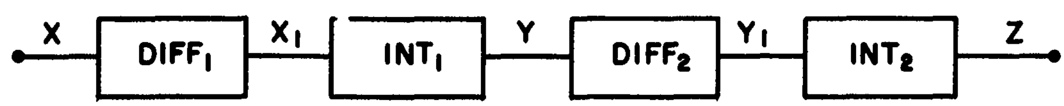
If the coefficients of like derivatives of y are equated, a system of m+n+1 simultaneous equations in the m+n+2 unknown $g_\alpha(t)$ and $h_\beta(t)$ are formed. Arbitrarily choosing $h_n(t) = 1$ (without loss of generality) allows these equations to be solved for the remaining m+n+1 unknowns. This operation shows the equivalence of Figures 3(a) and 3(b). Figure 3(b) can be further reduced to Figure 3(c) by combining the differential and integral operators and the resulting equation is:

$$\sum_{\beta=o}^{n} \sum_{k=o}^{m} \sum_{f=o}^{\beta} \binom{\beta}{f} h_\beta(t) \frac{d^{(\beta-f)} c_k(t)}{dt^{(\beta-f)}} \frac{d^{(k+f)} z}{dt^{(k+f)}} =$$

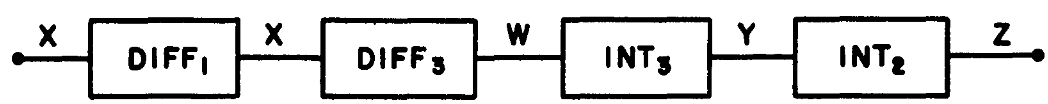


BLOCK DIAGRAM OF DIFFERENTIAL EQUATION

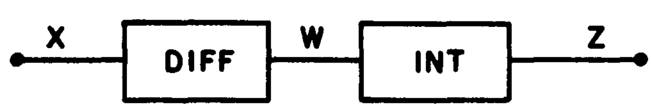
FIGURE 2



(a) PRODUCT OF TWO DIFFERERENTIAL EQUATIONS



(b) EQUIVALENT BLOCK DIAGRAM OF FIGURE 7(a)



(c) EQUIVALENT BLOCK DIAGRAM OF FIGURE 7(b)

MULTIPLICATION OF TWO DIFFERENTIAL EQUATIONS

FIGURE 3

$$= \sum_{\alpha=o}^{m} \sum_{j=o}^{n} \sum_{r=o}^{\alpha} \binom{\alpha}{r} g_\alpha(t) \frac{d^{(\alpha-r)} b_j(t)}{dt^{(\alpha-r)}} \frac{d^{(j+r)} x}{dt^{(j+r)}} \tag{18}$$

where x is the system input and z the output. Multiplication of two differential equations defined by Equations (13) and (14) is then defined by the steps indicated in Equations (15), (16), (17), and (18) and is denoted symbolically by

$$A\ B = C \tag{19}$$

where A corresponds to (13), B to (14), and C to (18).

The Unity Element. The unity element is defined as that element which when applied to a function leaves the function unchanged. In differential equation algebra, the unity element is any differential equation of the form

$$\sum_{i=o}^{n} a_i(t) \frac{d^i x}{dt^i} = \sum_{i=o}^{n} a_i(t) \frac{d^i y}{dt^i} \tag{20}$$

where x is the input and y is the output. Since the initial conditions are assumed zero, y=x and this equation indeed represents a unity element which will be denoted I.

The Multiplicative Inverse. A multiplicative inverse is defined as a differential equation (which will be denoted A^{-1}) which has the property that if it is multiplied by the differential equation A, a unity element is produced. Symbolically this is denoted

$$A^{-1}A = A\,A^{-1} = I \tag{21}$$

It was shown in Reference 8 that if A is represented by the differential equation

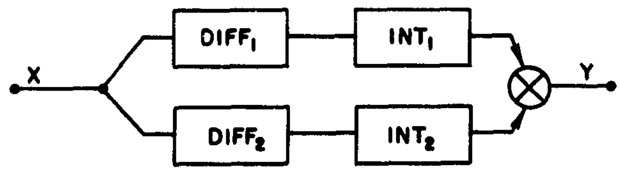
$$\sum_{i=o}^{n} a_i(t) \frac{d^i y}{dt^i} = \sum_{j=o}^{n} b_j(t) \frac{d^j x}{dt^j} \tag{22}$$

where x is the input and y is the output, then A^{-1} is represented by the differential equation

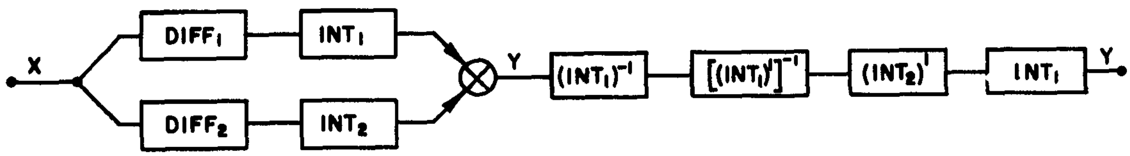
$$\sum_{j=o}^{n} b_j(t) \frac{d^j x}{dt^j} = \sum_{i=o}^{n} a_i(t) \frac{d^i y}{dt^i} \tag{23}$$

where x is the output and y is the input.

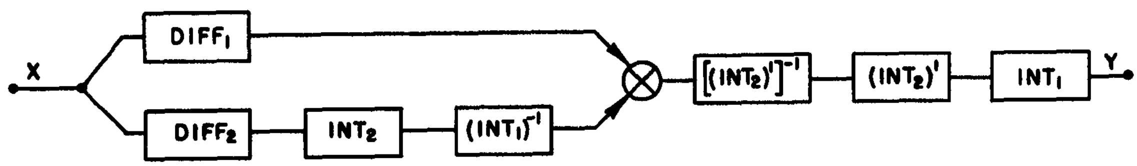
Addition of Two Differential Equations. Addition of two differential equations is represented symbolically in Figure 4(a). As in the case of multiplication, the differential equations have been divided into differential and integral operators. The addition is performed step-by-step as indicated by the block diagrams in steps indicated in Figure 4. In step 1 the original system is multiplied by the cascade combination of two differential



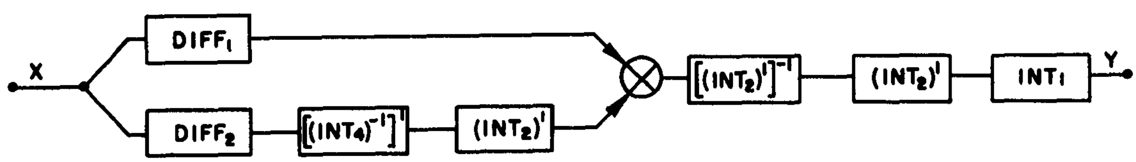
4 (a) ORIGINAL SYSTEM



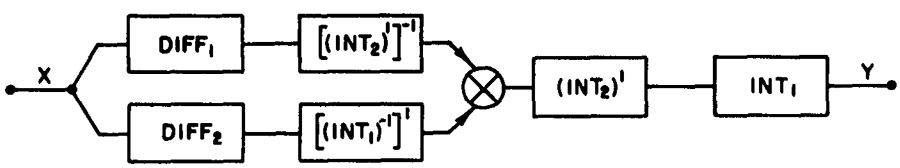
4 (b) STEP 1



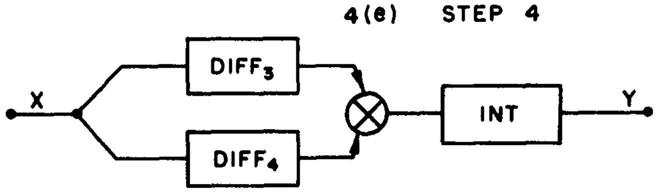
4 (c) STEP 2



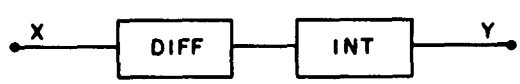
4 (d) STEP 3



4 (e) STEP 4



4 (f) STEP 5



4 (g) STEP 6

ADDITION OF DIFFERENTIAL EQUATIONS

FIGURE 4

operators denoted $(\mathrm{Int}_1)^{-1}$ and $\left[(\mathrm{Int}_2)'\right]^{-1}$ and two integral operators Int_1 and $(\mathrm{Int}_2)'$ which are their respective multiplicative inverses. This combination then represents a unity element. Since the system is linear $(\mathrm{Int}_1)^{-1}$ can be moved to the left of the summing junction, thus eliminating Int_1 from the upper element as shown in Figure 4(c). Now the integral operator Int_2 and the differential operator $(\mathrm{Int}_1)^{-1}$ are multiplied and the product separated into its new differential and integral operators as indicated in step 3, where the new differential operator is denoted $\left[(\mathrm{Int}_1)^{-1}\right]'$ and the new integral operator is denoted $(\mathrm{Int}_2)'$. $(\mathrm{Int}_2{}')^{-1}$ is now moved to the left of the summing junction thus cancelling $(\mathrm{Int}_2)'$. To the left of the summing junction there are now only differential operators and to the right only integral operators. These can then be combined as indicated in steps 5 and 6, and thus addition of differential equations has been defined.

The Zero Element. The zero element in differential equation algebra is defined as any differential equation whose output is always identically zero. The zero element is denoted 0.

The Additive Inverse. The additive inverse of a differential equation A is defined as that differential equation B which when added to A produces a zero element, i.e.,

$$A + B = 0 \tag{24}$$

It is quite obvious that if A is the differential equation

$$\sum_{i=o}^{n} a_i(t) \frac{d^i y}{dt^i} = \sum_{j=o}^{n} b_j(t) \frac{d^j x}{dt^j} \tag{25}$$

where x is the input and y is the output, then B has the form

$$\sum_{i=o}^{n} a_i(t) \frac{d^i y}{dt^i} = -\sum_{j=o}^{n} b_j(t) \frac{d^j x}{dt^j} \tag{26}$$

Multiplication of a Differential Equation by a Scalar. Multiplication of a differential equation by a scalar can be considered to be a degenerate case of multiplication of two differential equations where one of the differential equations is the degenerate equation

$$y = p(t)x \tag{27}$$

in which x is the input and y is the output. The techniques developed for multiplying differential equations are then applicable to multiplication of a differential equation by a scalar.

In addition to the above mentioned properties of differential equation algebra, the following laws hold:

(1) Addition is commutative, i.e., $A+B = B+A$

(2) Addition is associative, i.e., $A+(B+C) = (A+B)+C$

(3) Multiplication is not commutative, i.e., $A\ B \neq B\ A$
(In the stationary case multiplication is commutative.)

(4) Multiplication is associative, i.e., $A\ (B\ C) = (A\ B)\ C$

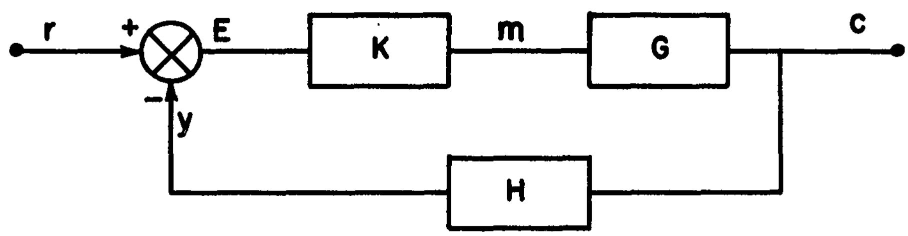
(5) Distributivity is valid, i.e., $A\ (B+C) = A\ B + A\ C$

In this section an algebra for differential equations has been discussed. With the aid of this algebra it is shown in the next section how linear system synthesis can be carried out operating only on differential equations.

SYNTHESIS OF A DIFFERENTIAL EQUATION AS A FEEDBACK SYSTEM

Once the differential equation algebra of the preceding section has been developed, its application to the synthesis problem is as straightforward conceptually as is the application of Laplace transform algebra in the synthesis of linear stationary systems. The advantage of such an algebra is that all of the manipulative operations can be performed symbolically and the numerical details carried through only at the end of the process.

To describe the technique of synthesis, the feedback configuration in Figure 5 will be examined. In this figure r is the input, c is the output, and K, G, and H are differential equations. m and ϵ are intermediate variables in the system. In the following the



GENERAL FEEDBACK CONFIGURATION

FIGURE 5

operation (x) represents the operation which a differential equation performs on a variable to produce a new variable. Let the relationship between the input r(t) and the output c(t) be

$$c = Wxr \tag{28}$$

where W is the desired overall differential equation. From the figure it is seen that

$$y = Hxc \tag{29}$$

and $\epsilon = r - y = r - Hxc$ (30)

Since $C = GKx\epsilon$ (31)

then $r = Ix\epsilon + HGKx\epsilon = [I + HGK]\ x\epsilon$ (32)

by virtue of the fact that

$$Ix\epsilon = \epsilon \tag{33}$$

Applying the multiplicative inverse of (I + HGK) to both sides of (32) the equation

$$\epsilon = \left[I + HGK\right]^{-1} xr \tag{34}$$

results. Then since

$$c = GK\, x\, \epsilon \tag{35}$$

substitution of (34) into (35) produces the equation

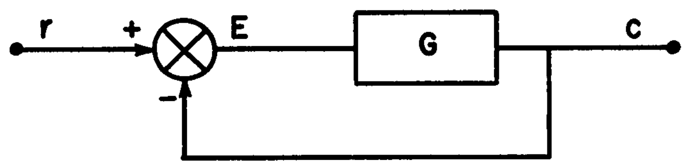
$$c = GK \left[I + HGK\right]^{-1} xr \quad . \tag{36}$$

Comparison of Equations (28) and (36) reveals that

$$W = GK \left[I + HGK\right]^{-1} \tag{37}$$

Equation (37) may be considered the fundamental relationship for Figure 5. Two special cases of the configuration in Figure 5 will now be considered.

Synthesis of a Feedback System Unconstrained by a Fixed Plant. Suppose that a given overall differential equation W is to be realized as a feedback system with a unity feedback, i.e., it is to have the configuration in Figure 6. The problem then is to determine



UNITY GAIN FEEDBACK SYSTEM

FIGURE 6

G in terms of W. Since in this case

$$K = H = I \tag{38}$$

Equation (37) reduces to

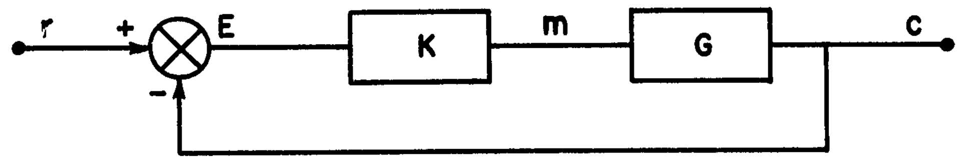
$$W = G(I+G)^{-1} \tag{39}$$

Solving Equation (39) for G the relationship

$$G = W(I - W)^{-1} \tag{40}$$

is obtained. Therefore a given differential equation W can be synthesized as a unity feedback system with G as the feedforward element through Equation (40). The actual differential equation of G is obtained by performing the operations indicated in Equation (40).

Synthesis of a System Constrained by a Fixed Plant. In this case let W be the differential equation of the overall system of the configuration in Figure 7. G is the differential equation of the known fixed plant. The problem is to find the differential equation of a



FEEDBACK SYSTEM WITH FIXED PLANT

FIGURE 7

suitable compensation network K such that this system has the desired overall response indicated by W. Setting H = I in Equation (37) and solving for K, the relationship

$$K = G^{-1}\left[I - W\right]^{-1} W \tag{41}$$

is formed. Performing the operations indicated symbolically in Equation (41), the differential equation for K is obtained.

Example: As an example of the process described above consider the following problem. Suppose that G in Figure 7 is a fixed plant described by the differential equation

$$\frac{d^2c}{dt^2} + \frac{dc}{dt} + e^{-t}c = m(t) \tag{42}$$

and it is desired to choose K such that the overall differential equation W has the form

$$\frac{d^2c}{dt^2} + 2\frac{dc}{dt} + c = r(t) \tag{43}$$

which is obviously stationary. Because of the stationarity of W the differential equation I-W is easily obtained by the method of addition of differential equations as:

$$\frac{d^2z}{dt^2} + 2\frac{dz}{dt} = \frac{d^2c}{dt^2} + 2\frac{dc}{dt} + c \tag{44}$$

where z is the input and c is the output. The differential equation $(I-W)^{-1}W$ can then be formed and is given by:

$$\frac{d^2c}{dt^2} + 2\frac{dc}{dt} = \epsilon(t) \tag{45}$$

The differential equation for K is finally determined by forming the product

$$K = G^{-1}(I-W)^{-1}W \tag{46}$$

and the equation for this compensation network is found to be:

$$\frac{d^2\epsilon}{dt^2} + \frac{2+3e^{-t}}{1+e^{-t}} \frac{d\epsilon}{dt} + (1+e^{-t})\epsilon = \frac{d^2m}{dt^2} + \frac{3+4e^{-t}}{1+e^{-t}} \frac{dm}{dt} + \frac{2+3e^{-t}}{1+e^{-t}} m \qquad (47)$$

Thus the compensating network is completely specified by Equation (47). This network can be synthesized by analog computer elements in a relatively straightforward manner by the techniques in Reference 15.

CONCLUSIONS

In this paper and a companion paper 18 a technique for synthesizing linear nonstationary feedback systems has been developed. For the synthesis of systems which receive inputs which are polynomials in time, the technique is complete, i.e., it includes both the approximation problem and the synthesis problem. In any event once the overall system weighting function or overall system differential equation has been specified, the synthesis technique leads to an exact synthesis of the system.

Also included in this paper is a discussion of an algebra of differential equations which is valuable to problems in both analysis and synthesis. This algebra allows manipulation of linear, nonstationary systems in a more logical manner than has been possible heretofore.

The synthesis technique has been applied to a simple single loop system in this report. There is no reason, however, for its not being applied to both multiple-loop and multiple-variable systems with equal facility.

In conclusion, as a result of a review of the literature, it appears that this is the first general and complete synthesis technique which is applicable to a wide class of linear, nonstationary feedback systems.

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